VCU, Department of Computer Science

CMSC 302

Relations 2

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Reminder About

Basic Properties of Relations

Definitions:

A relation R on a set A is called **reflexive** if (a, a)∈R for **every** element a∈A.

A relation R on a set A is called **symmetric** if (b, a)∈R whenever (a, b)∈R **for all** a, b∈A.

A relation R on a set A is called **transitive** if whenever (a, b)∈R and (b, c)∈R, then (a, c)∈R for a, b, c ∈ A.

Representing Relations

- We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs (digraphs)**.
- If R is a relation from A = $\{a_1, a_2, \ldots, a_m\}$ to B = $\{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with
- $m_{ij} = 1$, if $(a_i, b_j) \in R$, and
- $m_{ij} = 0$, if $(a_i, b_j) \notin R$.
- Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

Representing Relations

- **Example:** How can we represent the relation R defined between the set A{1, 2, 3} and set B{1, 2} where R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

- **Solution:** The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations

•What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?

•They are **square** matrices.

•What do we know about matrices representing **reflexive** relations?

•All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations

•What do we know about the matrices representing **symmetric relations**?

•These matrices are symmetric, that is, $M_R = (M_R)^t$.

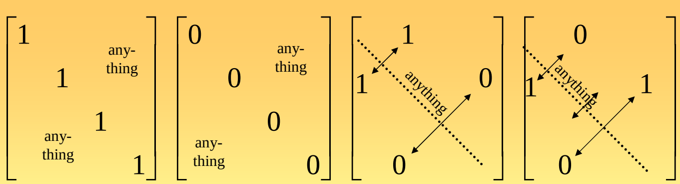
$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

•symmetric matrix, symmetric relation.

•non-symmetric matrix, non-symmetric relation.

Zero-One Reflexive, Symmetric

- Terms: *Reflexive*, *non-reflexive, irreflexive*, *symmetric, asymmetric, and antisymmetric*.
  - These relation characteristics are very easy to recognize by inspection of the zero-one matrix.



Reflexive: all 1's on diagonal

Irreflexive: all 0's on diagonal

Symmetric: all identical across diagonal

Antisymmetric: all 1's are across from 0's

Representing Relations

•The Boolean operations **join** and **meet** can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

•To obtain the **join** of two zero-one matrices, we apply the Boolean "or" function to all **corresponding** elements in the matrices.

•To obtain the **meet** of two zero-one matrices, we apply the Boolean "and" function to all **corresponding** elements in the matrices.

Representing Relations Using Matrices

•**Example:** Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

•What are the matrices representing R∪S and R∩S?

•**Solution:** These matrices are given by

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R \cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Representing Relations Using Matrices

-Do you remember the **Boolean product** of two zero-one matrices?

-Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.

-Then the **Boolean product** of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where

-$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

-$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

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Representing Relations Using Matrices

-Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C, respectively.

-**Remember:** For $M_C = M_A \circ M_B$ we have:

-$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

-In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

-Therefore, $C = B \circ A$ (**composite** of A and B).

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Representing Relations Using Matrices

-This gives us the following rule:

-$M_{B \circ A} = M_A \circ M_B$

-In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.

-Analogously, we can find matrices representing the **powers of relations**:

-$M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

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Representing Relations Using Matrices

•**Example:** Find the matrix representing R², where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

•**Solution:** The matrix for R² is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

How did we get this one? Well, …

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} (0\wedge 0)\vee(1\wedge 0)\vee(0\wedge 1) & 1 & 1 \\ 1 & (0\wedge 1)\vee(1\wedge 1)\vee(1\wedge 0) & 1 \\ 0 & 1 & (1\wedge 0)\vee(0\wedge 1)\vee(0\wedge 0) \end{bmatrix}$$

Representing Relations Using Digraphs

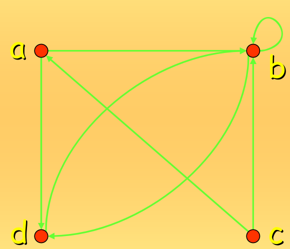
•**Definition:** A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

•The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

•We can use arrows to display graphs.

Representing Relations Using Digraphs

•**Example:** Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



•An edge of the form (b, b) is called a **loop**.

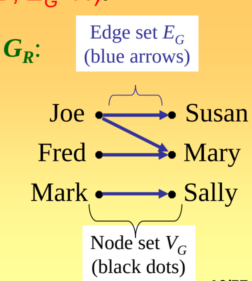
Using Directed Graphs

- Def. A *directed graph* or *digraph* $G=(V_G,E_G)$ is a set V_G of *vertices (nodes)* with a set $E_G \subseteq V_G \times V_G$ of *edges (arcs,links)*. Visually represented using dots for nodes, and arrows for edges. Notice that a relation $R{:}A\times B$ can be represented as a graph $G_R=(V_G=A\cup B,\ E_G=R)$.

Matrix representation $\mathbf{M}_R$:

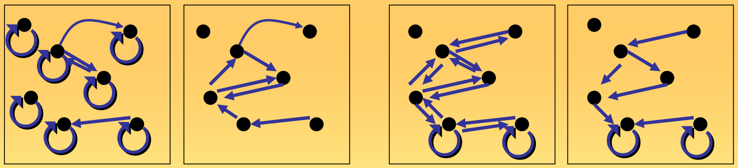
	Susan	Mary	Sally
Joe	1	1	0
Fred	0	1	0
Mark	0	0	1

Graph representation G_R:



Digraph Reflexive, Symmetric

- It is extremely easy to recognize the reflexive/irreflexive/ symmetric/antisymmetric properties by graph inspection.



Reflexive: Every node has a self-loop	Irreflexive: No node links to itself	Symmetric: Every link is bidirectional	Antisymmetric: No link is bidirectional

These are asymmetric & non-antisymmetric

These are non-reflexive & non-irreflexive

Representing Relations Using Digraphs

•Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs (a, b)∈R as its edges.

•Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

•This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.

This then means that **digraphs are sets**, and that **all the set operations apply**. We'll use it in **closures** which come next!

Closures of Relations, or Relational Closures

- Three types we will study
  - Reflexive
    - Easy
  - Symmetric
    - Easy
  - Transitive
    - Hard

Closures of Relations

- Def. For any property *X*, the "*X* closure" of a set *A* is defined as the "**smallest**" superset of *A* that has the given property.

  The ***reflexive closure*** of a relation *R* on *A* is obtained by adding (*a*, *a*) to *R* for each *a*∈*A*. *i.e.*, it is $R \cup I_A$

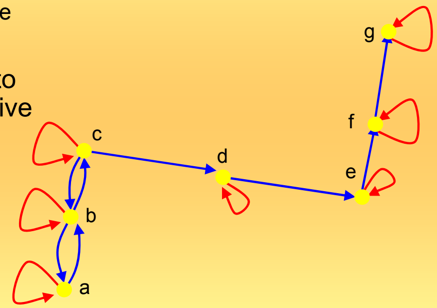
  The ***symmetric closure*** of *R* is obtained by adding (*b*,*a*) to *R* for each (*a*, *b*) in *R*. *i.e.*, it is $R \cup R^T$(note in book is R^{-1} used)

- The ***transitive closure*** or ***connectivity relation*** of *R* is obtained by repeatedly adding (*a*,*c*) to *R* for each (*a*,*b*),(*b*,*c*) in *R*., *i.e.*, it is

$$R^* = \bigcup_{n \in \mathbf{Z}^+} R^n \quad = R \vee R^2 \vee R^3 \cdots \vee R^{n-1} \vee R^n$$

Reflexive closure

- Consider a relation R:
  - Note that it is not reflexive
- We want to add edges to make the relation reflexive
- By adding those edges, we have made a non-reflexive relation R into a reflexive relation
- This new relation is called the **reflexive closure** of R

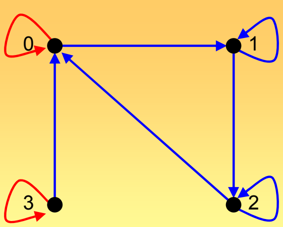


Reflexive closure

- In order to find the reflexive closure of a relation R, **we add a loop at each node that does not have one**
- The reflexive closure of R is $R \cup \Delta$
  - Where $\Delta = \{ (a, a) \mid a \in R \}$
    - Called the "diagonal relation"
  - **With matrices**, we set the diagonal to all 1's, meaning $\Delta = diag([1\ 1\ 1\ \ldots\ 1])$

Reflexive closure example

- Let R be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the reflexive closure of R?
- We add all pairs of edges (a,a) that do not already exist



We add edges:
(0,0), (3,3)

Reflexive closure example with matrices

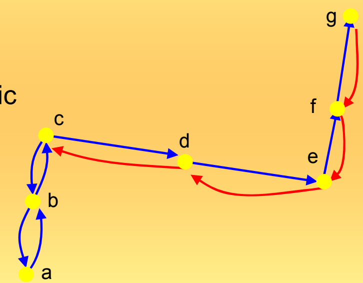
- Let R be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the reflexive closure of R?
- We 'add' a diagonal matrix with ones, called also identity matrix

$$R = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \cup \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

We add entries:
(0,0), (3,3)

Symmetric closure

- Consider a relation *R*:
  - Note that it is not symmetric
- We want to add edges to make the relation symmetric
- By adding those edges, we have made a non-symmetric relation *R* into a symmetric relation



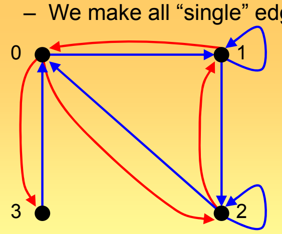
- This new relation is called the **symmetric closure** of *R*

Symmetric closure

- In order to find the symmetric closure of a relation *R*, we add an edge from *a* to *b*, where there is already an edge from *b* to *a*
- The symmetric closure of *R* is $R \cup R^T$
  - If *R* = { (*a*,*b*) | … }
  - Then R^T = { (*b*,*a*) | … }

Symmetric closure example

- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of *R*?
- We add all pairs of edges (*a*,*b*) where (*b*,*a*) exists
  - We make all "single" edges into anti-parallel pairs



We add edges:
(0,2), (0,3)
(1,0), (2,1)

Symmetric closure example with matrices

- Let *R* be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of *R*?
- We add all pairs of edges (*a*,*b*) where (*b*,*a*) exists
  - We make all "single" edges into anti-parallel pairs

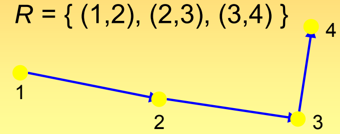
$$R = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \cup \begin{vmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

We add edges:
(0,2), (0,3), (1,0), (2,1)

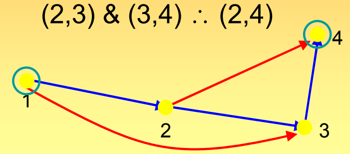
Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure
- First take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*
- But there is a path from 1 to 4 with no edge!

R = { (1,2), (2,3), (3,4) }

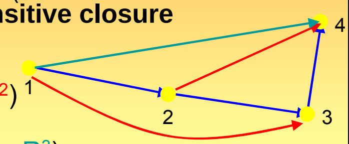


(1,2) & (2,3) ∴ (1,3)
(2,3) & (3,4) ∴ (2,4)



Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure
- Second take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*
  - Repeat this step until no new edges are added to the relation
- We will study **matrix algorithm** (out of few different ways) for **determining the transitive closure**



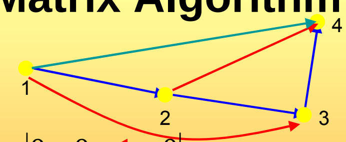
- red means added on the first repeat (that will be in R^2)
- teal means added on the second repeat (that will be in R^3)

Connectivity relation

- *R* contains edges between all the nodes reachable via 1 edge
- $R \circ R = R^2$ contains edges between nodes that are reachable via 2 edges in *R* (first repeat)
- $R^2 \circ R = R^3$ contains edges between nodes that are reachable via 3 edges in *R* (second repeat)
- R^n = contains edges between nodes that are reachable via *n* edges in *R*
- R^* contains edges between nodes that are reachable via any number of edges (i.e. via any path) in *R*
  - Rephrased: R^* contains all the edges between nodes *a* and *b* when is a path of length at least 1 between *a* and *b* in *R*
- ***R* is the transitive closure of R***
  - The definition of a transitive closure is that there are edges between any nodes (*a*,*b*) that contain a path between them

Transitive closure – Matrix Algorithm

R = { (1,2), (2,3), (3,4) } → M_R



$$M_{R^2} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} \odot \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$M_{R^3} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \odot \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

… it continues on slide 35

Finding the transitive closure with matrices

- Let $\mathbf{M}_R$ be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is:

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}$$

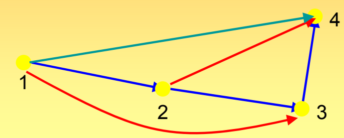
Nodes reachable with one application of the relation

Nodes reachable with two applications of the relation

Nodes reachable with *n* applications of the relation

Continuation of slide 33

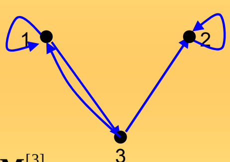
- Now you prove that $\mathbf{M}_{R^*} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$



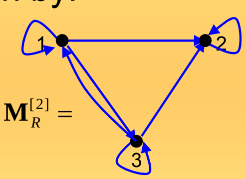
Sample questions

- Find the zero-one matrix of the transitive closure of the relation R given by:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



$\mathbf{M}_R^{[2]} =$

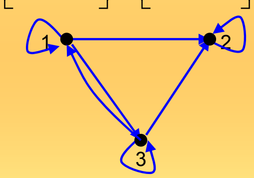


$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}$$

$$\mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Sample questions

$$\mathbf{M}_R^{[3]} = \mathbf{M}_R^{[2]} \odot \mathbf{M}_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Transitive closure algorithm

- What we did (or rather, could have done):
  - Compute the next matrix $\mathbf{M}_R^{[i]}$, where $1 \leq i \leq n$
  - Do a **Boolean join** with the previously computed matrix
- For our example:
  - Compute $\mathbf{M}_R^{[2]} = \mathbf{M}_R \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R$ to yield $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$
  - Compute $\mathbf{M}_R^{[3]} = \mathbf{M}_R^{[2]} \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$ from above

Transitive closure algorithm

procedure *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
  A := $\mathbf{M}_R$
  B := **A**
  for i := 2 **to** n
  begin
    A := **A** $\odot$ $\mathbf{M}_R$
    B := **B** $\vee$ **A**
  end { **B** is the zero-one matrix for R^* }

More transitive closure algorithms

- More efficient algorithms exist, such as Warshall's algorithm
  - We won't be studying it in this class

Equivalence Relations

•**Equivalence relations** are used **to relate objects that are similar in some way**.

•**Definition:** A relation on a set A is called an **equivalence** relation if it is **reflexive, symmetric,** and **transitive.**

•Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations

•Since R is **reflexive**, every element is equivalent to itself. (For every $a \in S$, *aRa*).

•Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a. (If *aRb then bRa*)

•Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent. (If *aRb and bRc then aRc*).

•Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations

The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike". In fact, the relation "=" of equality on any set S is an equivalence relation; that is:

(1) $a = a$ for every $a \in S$.
(2) If $a = b$, then $b = a$.
(3) If $a = b$ and $b = c$, then $a = c$.

More equivalency:

(*a*) Consider the set L of lines in the Euclidean plane. The relation "is parallel to is an equivalence relation on L

(*b*) The classification of animals by species, that is, the relation "is of the same species as", is an equivalence relation on the set of animals.

(*c*) The relation $\subseteq$ of set inclusion is not an equivalence relation. It is reflexive and transitive, but it is not symmetric since $A \subseteq B$ does not imply $B \subseteq A$.

(*d*) Let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, written

$$a \equiv b \pmod{m}$$

if m divides $a - b$. For example, for $m = 4$ we have $11 \equiv 3 \pmod 4$ since 4 divides $11 - 3$, and $22 \equiv 6 \pmod 4$ since 4 divides $22 - 6$. This relation of congruence modulo m is an equivalence relation.

Proof that 'congruence modulo *m*' is an equivalence relation

$11 \equiv 3 \pmod 4$,

because it's reflexive $11 \equiv 11 \pmod 4$,

it's symmetric $3 \equiv 11 \pmod 4$,

and it is transitive

$11 \equiv 3 \pmod 4$ and $3 \equiv -1 \pmod 4$,

results into $11 \equiv -1 \pmod 4$.

Equivalence Relation More Examples

- "Strings *a* and *b* are the same length."(see next slide)
- "Integers *a* and *b* have the same absolute value."
- "Integers *a* and *b* have the same residue modulo *m*." (for a given *m*>1, see previous slide)

Equivalence Relations

•**Example:** Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if L(a) = L(b), where L (x) is the length of the string x.

Is R an equivalence relation?

•**Solution:**

- R is reflexive, because L(a) = L(a) and therefore aRa for any string a.
- R is symmetric, because if L(a) = L(b) then L(b) = L(a), so if aRb then bRa.
- R is transitive, because if L(a) = L(b) and L(b) = L(c), then L(a) = L(c), so aRb and bRc implies aRc.

•R is an equivalence relation.

Equivalence Classes

•**Definition:** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of **a**.

•The equivalence class of **a** with respect to R is denoted by $[a]_R$.

•When only one relation is under consideration, we will delete the subscript R and write **[a]** for this equivalence class.

•If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes

•**Example:** In the previous example (**strings of identical length**), what is the equivalence class of the word mouse, denoted by [mouse] ?

•**Solution:** [mouse] is the set of all English words containing five letters.

•For example, 'horse' would be a representative of this equivalence class.

Equivalence Classes

•**Theorem:** Let R be an equivalence relation on a set A. The following statements are equivalent:

- aRb
- [a] = [b]
- $[a] \cap [b] \neq \emptyset$

•**Definition:** A **partition** of a set S is a collection of **disjoint** nonempty subsets of S **that have S as their union**. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

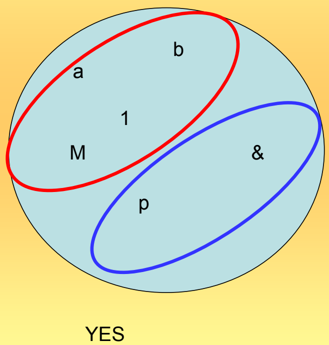
(*i*) $A_i \neq \emptyset$ for $i \in I$

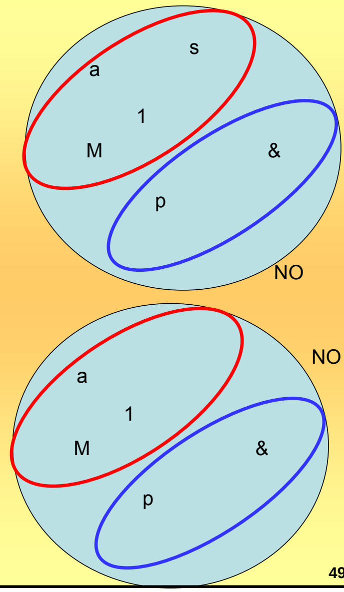
(*ii*) $A_i \cap A_j = \emptyset$, if $i \neq j$

(*iii*) $\cup_{i \in I} A_i = S$

Examples of partitions

S={a, b, M, p, 1, &}





Equivalence Classes

•**Examples:** Let S be the set {u, m, b, r, o, c, k, s}.
Do the following collections of sets partition S ?

•{{m, o, c, k}, {r, u, b, s}}	•yes.
•{{c, o, m, b}, {u, s}, {r}}	•no (k is missing).
•{{b, r, o, c, k}, {m, u, s, t}}	•no (t is not in S).
•{{u, m, b, r, o, c, k, s}}	•yes.
•{{b, o, o, k}, {r, u, m}, {c, s}}	•yes ({b,o,o,k} = {b,o,k}).
•{{u, m, b}, {r, o, c, k, s}, ∅}	•no (∅ not allowed).

Equivalence Classes

•**Theorem:** Let R be an equivalence relation on a set S.

•Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Classes

•**Example:** Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Sava lives in Belgrade.

•Let R be the **equivalence relation** {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Sava}.

•Then R = {(Frank, Frank), (Frank, Suzanne), (Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Sava, Sava)}. … it continues

Equivalence Classes

•Then the **equivalence classes** of R are:

•{{Frank, Suzanne, George}, {Stephanie, Max}, {Sava}}.

•This is a **partition** of P.

•The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Equivalence Classes

•**Another example:** Let R be the relation {(a, b) | a ≡ b (mod 3)} on the set of integers.

•Is R an equivalence relation?

•Yes, R is reflexive, symmetric, and transitive.

•What are the equivalence classes of R ?

•{{…, -6, -3, 0, 3, 6, …},
{…, -5, -2, 1, 4, 7, …},
{…, -4, -1, 2, 5, 8, …}}

Quick survey

- I understood the material in this slide set…

a) Very well, or close
b) With some review, I'll be good
c) Not really
d) Not at all

Quick survey

The pace of the lecture for this slide set was…

a) Fast
b) About right
c) A little slow
d) Too slow

Quick survey

- How interesting was the material in this slide set? Be honest!

a) Wow! That was cooooOoooool!

b) Somewhat interesting

c) Rather boring

d) zzzzzzzzzzzzzzzzzzzzzzzZZ